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ON WEAK CONVERGENCE OF INTEGRAL FUNCTIONALS OF STOCHASTIC PROCESSES WITH APPLICATIONS TO PROCESSES TAKING PATHS IN L_p^E

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The weak convergence of certain functionals of a sequence of stochastic processes is investigated. The functionals under consideration are of the form $f_\phi(x) = \int \phi(t, x(t))\mu(\mathrm{d}t)$. The main result is as follows: If a sequence $\{\xi_n: n \in \mathbb{N}\}$ is weakly tight in a certain sense, and, in addition, the finite dimensional distributions of the processes converge weakly, then this implies weak convergence of the functionals $(f_{\phi_1}(\xi_n), \dots, f_{\phi_m}(\xi_n))$ to $(f_{\phi_1}(\xi_0), \dots, f_{\phi_m}(\xi_0))$. Necessary and sufficient conditions for weak tightness are stated and applications of the results to the case of L_p^E-valued stochastic processes are given. In particular it is shown that the usual tightness condition for weak convergence of such processes can be considerably weakened.

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weak convergence * tightness * integral functionals * finite dimensional distributions * stochastic processes * L_p^E-spaces

1. Introduction

Many applications of the theory of weak convergence of stochastic processes involve functionals in the form of integrals: I. I. Gikhman and A. V. Skorokhod [10] state conditions in order for the distributions of $\int_0^1 \xi_n(t)\,\mathrm{d}t$ to weakly converge to $\int_0^1 \xi_0(t)\,\mathrm{d}t$, where $\{\xi_n: n \in \mathbb{N}_0\}$ is a sequence of measurable stochastic processes on $[0, 1]$. L.S. Grinblat has shown in [11] that these conditions essentially suffice to obtain weak convergence for all continuous functionals on $\mathscr{L}_p$. More recently, Grinblat [12, 13] has found additional conditions, some in a more general setting. A.A. Borovkov and E.A. Pecherskii [3] investigate functionals of the form $f(x) = \int_0^1 \phi(x(t))\,\mathrm{d}t$ for certain continuous functions ϕ on $\mathbb{R}$. Via the theory of σ-topological spaces they arrive at a new set of conditions for the weak convergence of $f(\xi_n)$ to $f(\xi_0)$. Still other conditions for functionals of this kind, this time for ϕ in $\mathscr{L}_2$, are given by S.M. Berman [1].

In the present paper we propose to unify these results in a considerably more general framework. We regard measurable processes which take values in a completely regular Hausdorff space, and functionals of the form $f_\phi(x) := \int \phi(t, x(t))\mu(\mathrm{d}t)$, where ϕ is continuous in its second component and μ is a σ-finite measure on some measurable space. Our main result is as follows: If a sequence $\{\xi_n: n \in \mathbb{N}_0\}$ of measurable processes is weakly tight in a certain sense, and, in addition, the finite dimensional distributions of the processes converge weakly, then this implies weak convergence of the functionals $(f_{\phi_1}(\xi_n), \ldots, f_{\phi_m}(\xi_n))$ to $(f_{\phi_1}(\xi_0), \ldots, f_{\phi_m}(\xi_0))$ (Section 2). Necessary and sufficient conditions for weak tightness are stated in a subsequent section. Important applications of the main theorem are studied in Section 4. There we first show that, for processes with paths in $\mathscr{L}_p^E$, E a Banach space, the usual tightness condition which is necessary and sufficient for weak convergence (cf. [4, Theorem 2]) can be considerably weakened, and, as a consequence, generalizations of some of the results by Grinblat referred to above can be derived. A further application concerns functions of the form $f(x) = \int \langle x, g \rangle \, d\mu$, $g \in L_q^{E'}$; here we are able to generalize a result by Berman mentioned above. A final application deals with quantile processes in the Hilbert space $\mathscr{L}_2(0, 1)$. Under weak conditions we can show, supplementary to a theorem by Mason (cf. [14, Theorem 3]), that the quantile processes converge weakly to a Brownian bridge.

2. Main result

In the sequel we employ the following notation.

- $(\Omega, \mathscr{A}, \mathbb{P})$ for a probability space;
- $(T, \mathscr{B}, \mu)$ for a σ-finite measure space;
- S for a completely regular Hausdorff space with Borel σ-field $\mathscr{B}_S = \mathscr{B}(S)$;
- $C(S)[C_b(S)]$ for the space of continuous [continuous and bounded] real-valued functions;
- $(E, |\cdot|)$ for a Banach space with dual E';
- $\mathscr{L}_p^E(\mu) = \mathscr{L}_p^E(T, \mathscr{B}, \mu)$ for the space of p-integrable E-valued functions with seminorm $\|x\|_p^E := (\int |x(t)|^p \mu(dt))^{1/p}$;
- $L_p^E(\mu) = L_p^E(T, \mathscr{B}, \mu)$ for the corresponding Banach space (of equivalence classes modulo μ-null functions);
- $\mathscr{L}_p(\mu) := \mathscr{L}_p^{\mathbb{R}}(\mu)$ and $L_p(\mu) := L_p^{\mathbb{R}}(\mu)$;
- $\mathscr{L}_p^+(\mu) := \{x \in \mathscr{L}_p(\mu) : x \geq 0\}$;
- $a^+ := \max\{0, a\}$ for $a \in \mathbb{R}$.

For $\mathscr{A}$-$\mathscr{B}_S$-measurable maps $X_n^i : \Omega \to S$, $i = 1, \ldots, k$; $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we write $(X_n^1, \ldots, X_n^k) \Rightarrow (X_0^1, \ldots, X_0^k)$ iff $\int g(X_n^1, \ldots, X_n^k) \, d\mathbb{P} \to \int g(X_0^1, \ldots, X_0^k) \, d\mathbb{P}$ for each $\bigotimes_1^k \mathscr{B}_S$-measurable $g \in C_b(S^k)$. Note that for $k = 1$ or S separable metric this is usual weak convergence.

Now, let Φ be a set of $\mathscr{B} \otimes \mathscr{B}_S$-measurable functions $\phi : T \times S \to \mathbb{R}$ with $\phi(t, \cdot) \in C(S)$ for all $t \in T$ and M a set of $\mathscr{B}$-$\mathscr{B}_S$-measurable functions $x : T \to S$ with

$$\int |\phi(t, x(t)| \mu(dt) < +\infty, \quad \phi \in \Phi. \tag{2.1}$$

Then $f_\phi : M \to \mathbb{R}^1$, $\phi \in \Phi$, is defined by

$$f_\phi(x) = \int \phi(t, x(t)) \mu(dt), \quad x \in M. \tag{2.2}$$

Finally, let $\{\xi_n: n \in \mathbb{N}_0\}$be a sequence of $\mathscr{A}\otimes\mathscr{B}$-$\mathscr{B}_S$-measurable functions $\xi_n: \Omega \times T \to S$ with

$$\xi_n(\omega, \cdot) \in M, \quad \omega \in \Omega, \ n \in \mathbb{N}_0, \tag{2.3}$$

i.e. the ξ_n are measurable stochastic processes with paths in M. Note that for $\phi \in \Phi$ the map $(\omega, t) \to \xi_n^\phi(\omega, t) := \phi(t, \xi_n(\omega, t))$ is $\mathscr{A}\otimes\mathscr{B}$-measurable, and that by (2.1) and (2.3), for all $\omega \in \Omega$, $\hat{\xi}_n^\phi(\omega) := \xi_n^\phi(\omega, \cdot)$ is an element of $L_1(\mu)$. By a standard argument ξ_n^ϕ is $\mathscr{A}\otimes\mathscr{C}$-measurable for a suitable countably generated σ-algebra $\mathscr{C} \subset \mathscr{B}$ with $\mu \mid \mathscr{C}$ σ-finite. Thus $\hat{\xi}_n^\phi$ has values in the Polish space $L_1(T, \mathscr{C}, \mu) \subset L_1(\mu)$. Hence, by [4, Lemma 1], $\omega \to \hat{\xi}_n^\phi(\omega)$ is $\mathscr{A}$-$\mathscr{B}(L_1(\mu))$-measurable. This also holds for $\omega \to \hat{\xi}_n(\omega) := \xi_n(\omega, \cdot)$, where ξ_n takes paths in $\mathscr{L}_1(\mu)$.

Definition 1. The finite dimensional distributions (f.d.d.) of the processes ξ_n converge to those of ξ_0 almost everywhere (a.e.) iff there is $T_0 \in \mathscr{B}$ with $\mu(T - T_0) = 0$ such that for all $k \in \mathbb{N}$, $t_1, \dots t_k \in T_0$:

$$(\xi_n(\cdot, t_1), \dots, \xi_n(\cdot, t_k)) \Rightarrow (\xi_0(\cdot, t_1), \dots, \xi_0(\cdot, t_k)). \tag{2.4}$$

Definition 2. (a) A sequence $\{\xi_n: n \in \mathbb{N}\}$ of processes with paths in $\mathscr{L}_1(\mu)$ is called weakly tight iff for each $\varepsilon > 0$ there is $K \subset L_1(\mu)$ weakly compact, i.e. compact in the $\sigma(L_1(\mu), L_\infty(\mu))$-topology, such that $\inf_n \mathbb{P}\{\hat{\xi}_n(\omega) \in K\} > 1 - \varepsilon$. (b) A sequence $\{\xi_n: n \in \mathbb{N}\}$ of measurable processes with paths in M is called Φ-weakly tight iff for all $\phi \in \Phi$ the sequence $\{\xi_n^\phi: n \in \mathbb{N}\}$ is weakly tight.

Remarks 1. By the Dunford–Pettis compactness criterion (cf. [9, Lemma 83A]) a subset $K \subset L_1(\mu)$ is relatively weakly compact iff it is uniformly integrable, i.e. for each $\varepsilon > 0$ there is $h \in L_1^+(\mu)$ such that $\sup_{x \in K} \int (|x| - h)^+ \, \mathrm{d}\mu \leq \varepsilon$ (note that $(|x| - h)^+ = |x| - \min\{|x|, h\} \cdot 2$. Each image measure $\hat{\xi}_n^\phi(\mathbb{P})$ is concentrated on a separable subset of the Banach space $L_1(\mu)$.Hence $\hat{\xi}_n^\phi(\mathbb{P})$ is tight (even in the strong topology); in particular each ξ_n is always Φ-weakly tight.

Now we can state our main theorem concerning weak convergence of the random variables $f_\phi(\xi_n) = \int \phi(t, \xi_n(\cdot, t))\mu(\mathrm{d}t)$.

Theorem 1. *Let $\{\xi_n: n \in \mathbb{N}_0\}$ be a sequence of stochastic processes satisfying condition* (2.3). *If the finite dimensional distributions of ξ_n converge weakly to those of ξ_0 a.e. and if $\{\xi_n: n \in \mathbb{N}\}$ is Φ-weakly tight, then for all $\phi_1, \dots, \phi_m \in \Phi$,*

$$(f_{\phi_1}(\xi_n), \dots, f_{\phi_m}(\xi_n)) \Rightarrow (f_{\phi_1}(\xi_0), \dots, f_{\phi_m}(\xi_0)).$$

Proof. Let $\phi \in \Phi$ and $h \in \mathscr{L}_1^+(\mu)$. Define $\phi^h(t, s) := \max\{-h(t), \min\{h(t), \phi(t, s)\}\}$ and

$$\eta_n^h(\omega, \phi) := \int \phi^h(t, \xi_n(\omega, t))\mu(\mathrm{d}t), \quad n \in \mathbb{N}_0. \tag{2.5}$$

Clearly $-\|h\|_1 \le \eta_n^h \le \|h\|_1$. First we show $\eta_n^h(\cdot,\phi) \Rightarrow \eta_0^h(\cdot,\phi)$, i.e., for $g\in C_b(\mathbb{R})$,

$$\int g(\eta_n^h(\omega,\phi))\mathbb{P}(\mathrm{d}\omega) \to \int g(\eta_0^h(\omega,\phi))\mathbb{P}(\mathrm{d}\omega). \tag{2.6}$$

Since $|\eta_n^h| \le \|h\|_1$ and polynomials are dense in $C([-\|h\|_1, \|h\|_1])$ (Stone–Weierstrass), let without loss $g(u)=u^l$ for some $l\in\mathbb{N}$. By Fubini's Theorem we obtain

$$\begin{aligned}
&\int g(\eta_n^h(\omega,\phi))\mathbb{P}(\mathrm{d}\omega)\\
&\quad=\int (\eta_n^h(\omega,\phi))^l\mathbb{P}(\mathrm{d}\omega)\\
&\quad=\int\left[\int \phi^h(t_1,\xi_n(\omega,t_1))\mu(\mathrm{d}t_1)\cdots\int \phi^h(t_l,\xi_n(\omega,t_l))\mu(\mathrm{d}t_l)\right]\mathbb{P}(\mathrm{d}\omega)\\
&\quad=\int\cdots\int F_n(t_1,\dots,t_l)\mu(\mathrm{d}t_1)\cdots\mu(\mathrm{d}t_l),
\end{aligned}$$

where

$$F_n(t_1,\dots,t_l):=\int \phi^h(t_1,\xi_n(\omega,t_1))\cdots\phi^h(t_l,\xi_n(\omega,t_l))\mathbb{P}(\mathrm{d}\omega),\quad n\in\mathbb{N}_0.$$

Since $(u_1,\dots,u_l)\to\phi^h(t_1,u_1)\cdots\phi^h(t_l,u_l)$ is in $C_b(S^k)$ and $\bigotimes_1^k\mathscr{B}_S$-measurable for fixed $t_1,\dots,t_l\in T_0$, we obtain $F_n(t_1,\dots,t_l)\to F_0(t_1,\dots,t_l)$ by weak convergence of the f.d.d. Since $|F_n(t_1,\dots,t_l)|\le h(t_1)\cdots h(t_l)$,

$$\begin{aligned}
&\int\cdots\int F_n(t_1,\dots,t_l)\mu(\mathrm{d}t_1)\cdots\mu(\mathrm{d}t_l)\\
&\quad\to\int\cdots\int F_0(t_1,\dots,t_l)\mu(\mathrm{d}t_1)\cdots\mu(\mathrm{d}t_l)
\end{aligned}$$

by dominated convergence, whence (2.6). Similarly for $\phi_1,\dots,\phi_m\in\Phi$:

$$(\eta_n^h(\cdot,\phi_1),\dots,\eta_n^h(\cdot,\phi_m)) \Rightarrow (\eta_0^h(\cdot,\phi_1),\dots,\eta_0^h(\cdot,\phi_m)). \tag{2.7}$$

For $\phi\in\Phi$ let $\eta_n(\omega,\phi):=f_\phi(\xi_n)(\omega)=\int\phi(t,\xi_n(\omega,t))\mu(\mathrm{d}t)$. Fix $\phi_1,\dots,\phi_m\in\Phi$. By Φ-weak tightness and the Remark following Definition 2, for all $N\in\mathbb{N}$ there is a weakly compact subset K_N of $L_1(\mu)$ with

$$\sup_{n\in\mathbb{N}}\mathbb{P}\{\omega\in\Omega\colon \hat{\xi}_n^{\phi_i}(\omega)\in K_N,\ i=1,\dots,m\}\ge 1-\frac{1}{N}; \tag{2.8}$$

and thus a $h_N\in\mathscr{L}_1^+(\mu)$ such that $\sup_{x\in K_N}\int(|x|-h_N)^+\,\mathrm{d}\mu\le 1/N$. Without loss, h_N can be chosen with $h_N\uparrow\infty$ for $N\uparrow\infty$. By dominated convergence $\eta_0^{h_N}(\cdot,\phi_i)\to\eta_0(\cdot,\phi_i)$ pointwise, hence in particular

$$(\eta_0^{h_N}(\cdot,\phi_1),\dots,\eta_0^{h_N}(\cdot,\phi_m)) \Rightarrow (\eta_0(\cdot,\phi_1),\dots,\eta_0(\cdot,\phi_m)). \tag{2.9}$$

Now by [2, Theorem 4.2 and (2.7), (2.9)] we obtain

$$(\eta_n(\cdot,\phi_1),\ldots,\eta_n(\cdot,\phi_m)) \Rightarrow (\eta_0(\cdot,\phi_1),\ldots,\eta_0(\cdot,\phi_m)),$$

provided, for all $\varepsilon>0$,

$$\lim_{N\to\infty}\limsup_{n\to\infty}\mathbb{P}\{\max_{1\leq i\leq m}|\eta_n^{h_N}(\omega,\phi_i)-\eta_n(\omega,\phi_i)|\geq\varepsilon\}=0.$$

But this is a simple consequence of (2.8) and

$$\begin{aligned}|\eta_n^{h_N}(\omega,\phi)-\eta_n(\omega,\phi)| &\leq \int |\phi^{h_N}(t,\xi_n(\omega,t))-\phi(t,\xi_n(\omega,t))|\mu(\mathrm{d}t)\\ &= \int (|\phi(t,\xi_n(\omega,t))|-h_N(t))^+\mu(\mathrm{d}t),\end{aligned}$$

where, for the last equation, we have used $|\max\{-a,\min\{a,b\}\}-b|=(|b|-a)^+$ for $b\in\mathbb{R}$, $a\geq 0$. □

3. Necessary and sufficient conditions for Φ-weak tightness

In part A of this section equivalent conditions for Φ-weak tightness are stated, and, additionally, in part B, with a special regard to applications, two easily provable sufficient conditions are derived. Finally, part C presents a generalized form of a result by Borovkov/Pecherskii [3] which is an immediate consequence of our main theorem. Note that this is achieved without using the theory of σ-topological spaces.

Let $\phi\in\Phi$ and ξ_n, $n\in\mathbb{N}_0$, be given as in Section 2, and let us recall the notation $[\hat{\xi}_n^\phi(\omega)](t)=\xi_n^\phi(\omega,t)=\phi(t,\xi_n(\omega,t))$.

A. **Lemma 1.** *The following conditions are equivalent.*

$$\{\xi_n\colon n\in\mathbb{N}\}\ \textit{is}\ \Phi\textit{-weakly tight};\tag{3.1}$$

$$\forall\varepsilon>0,\phi\in\Phi:\quad \inf_{h\in\mathscr{L}_1^+(\mu)}\limsup_{n\to\infty}\mathbb{P}\left\{\int(|\hat{\xi}_n^\phi(\omega)|-h)^+\,\mathrm{d}\mu\geq\varepsilon\right\}=0;\tag{3.2}$$

$$\forall\varepsilon>0,\phi\in\Phi:\quad \inf_{h\in\mathscr{L}_1^+(\mu)}\limsup_{n\to\infty}\mathbb{P}\left\{\int_{\{|\hat{\xi}_n^\phi(\omega)|\geq h\}}|\hat{\xi}_n^\phi(\omega)|\,\mathrm{d}\mu\geq\varepsilon\right\}=0;\tag{3.3}$$

$$\forall\varepsilon>0,\phi\in\Phi:\quad \lim_{N\to\infty}\limsup_{n\to\infty}\mathbb{P}\left\{\int(|\hat{\xi}_n^\phi(\omega)|-N)^+\,\mathrm{d}\mu\geq\varepsilon\right\}=0\ \textit{and}$$

$$\inf_{B\in\mathscr{B},\mu(B)<\infty}\limsup_{n\to\infty}\mathbb{P}\left\{\int_{T-B}|\hat{\xi}_n^\phi(\omega)|\,\mathrm{d}\mu\geq\varepsilon\right\}=0.\tag{3.4}$$

(Here $\mathbb{P}\{\int(|\hat{\xi}_n^\phi(\omega)|-h)^+\,\mathrm{d}\mu\geq\varepsilon\}$ *is shorthand for* $\mathbb{P}\{\omega\in\Omega\colon\int(|\phi(t,\xi_n(\omega,t))|-h(t))^+\mu(\mathrm{d}t)\geq\varepsilon\}$. *This shorthand notation is extended to analogous cases.)*

Proof. Recall from Definition 2 and the subsequent Remark that $\{\xi_n: n\in\mathbb{N}\}$ is Φ-weakly tight iff for all $\eta>0$ and $\phi\in\Phi$ there is $K\subset L_1(\mu)$ such that for all $\varepsilon>0$ there is $h\in\mathscr{L}_1^+(\mu)$ with $\sup_{x\in K}\int(|x|-h)^+\,\mathrm{d}\mu<\varepsilon$ and $\inf_{n\in\mathbb{N}}\mathbb{P}\{\hat{\xi}_n^\phi(\omega)\in K\}\geq 1-\eta$. (3.1)$\Rightarrow$(3.2): Assume (3.1) and let $\varepsilon>0$, $\eta>0$, and $\phi\in\Phi$ be given. First choose K and then h as described above. Then

$$\sup_n \mathbb{P}\left\{\int(|\hat{\xi}_n^\phi(\omega)|-h)^+\,\mathrm{d}\mu\geq\varepsilon\right\}\leq\sup_n\mathbb{P}\{\hat{\xi}_n^\phi(\omega)\notin K\}\leq\eta$$

and (3.2) follows, even with sup instead of lim sup. (3.2)$\Rightarrow$(3.1): First we show that lim sup in (3.2) can be replaced by sup. For $\varepsilon>0$ and $\phi\in\Phi$ fixed, let $a_n(h):=\mathbb{P}\{\int(|\hat{\xi}_n^\phi(\omega)|-h)^+\,\mathrm{d}\mu\geq\varepsilon\}$. Then by (3.2) $\limsup a_n(h)\downarrow 0$ for h increasing to ∞, and, by the Remark following Definition 2, $a_n(h)\downarrow 0$ for each $n\in\mathbb{N}$. Hence $\sup_n a_n(h)\downarrow 0$. Now let $\varepsilon>0$, $\eta>0$, and $\phi\in\Phi$ be given. For each $k\in\mathbb{N}$ choose $h_k\in L_1^+(\mu)$ such that $\sup_n[\hat{\xi}_n^\phi(\mathbb{P})](B_k)\geq 1-\eta/2^{k+1}$, where $B_k:=\{x\in L_1(\mu): \||x|-h_k\|_1\leq 1/k\}$. (3.1) is fulfilled for $K:=\bigcap_1^\infty B_k$ and h_{k_0} with $1/k_0\leq\varepsilon$. (3.2)$\Leftrightarrow$(3.3): Note that for all $a,b\geq 0$ we have $a1_{\{a\geq b\}}\leq 2(a-b/2)^+$; hence

$$\int(|\hat{\xi}_n^\phi(\omega)|-h)^+\,\mathrm{d}\mu\leq\int_{\{|\hat{\xi}_n^\phi(\omega)|\geq h\}}|\hat{\xi}_n^\phi(\omega)|\,\mathrm{d}\mu\leq 2\int(|\hat{\xi}_n^\phi(\omega)|-h/2)^+\,\mathrm{d}\mu,$$

and equivalence of (3.2) and (3.3) follows. (3.4)$\Rightarrow$(3.2): Put $h:=N1_B$ for suitable $N\in\mathbb{N}$ and $B\in\mathscr{B}$ with $\mu(B)<\infty$. (3.2)$\Rightarrow$(3.4): Assume (3.2). Then the first relation follows from

$$\int(|\hat{\xi}_n^\phi(\omega)|-N)^+\,\mathrm{d}\mu\leq\int(|\hat{\xi}_n^\phi(\omega)|-h)^+\,\mathrm{d}\mu+\int(h-N)^+\,\mathrm{d}\mu,$$

and the second from

$$\int_{T-B}|\hat{\xi}_n^\phi(\omega)|\,\mathrm{d}\mu\leq\int(|\hat{\xi}_n^\phi(\omega)|-h)^+\,\mathrm{d}\mu+\int_{T-B}h\,\mathrm{d}\mu.\qquad\square$$

Remark. The first condition of (3.4) is satisfied, if $(\omega,t)\to\phi(t,\xi_n(\omega,t))$, $n\in\mathbb{N}$, is uniformly bounded and the second always holds true for bounded measure μ.

B. The following two conditions are stated in terms of the product measure $\mathbb{P}\otimes\mu$.

Lemma 2. *$\{\xi_n: n\in\mathbb{N}\}$ is Φ-weakly tight, if*

$$\forall\phi\in\Phi:\quad\inf_{h\in\mathscr{L}_1^+(\mathbb{P}\otimes\mu)}\ \limsup_{n\to\infty}\int(|\xi_n^\phi|-h)^+\,\mathrm{d}\mathbb{P}\otimes\mu=0.\tag{3.5}$$

Proof. Let $g\in\mathscr{L}_1^+(\mu)$ and $h\in\mathscr{L}_1^+(\mathbb{P}\otimes\mu)$. Put $\tilde{g}(\omega,t):=g(t)$. From the Kolmogorov-Markov inequality we get

$$\mathbb{P}\left\{\int(|\hat{\xi}_n^\phi(\omega)|-g)^+\,\mathrm{d}\mu\geq\varepsilon\right\}\leq\frac{1}{\varepsilon}\int(|\xi_n^\phi|-\tilde{g})^+\,\mathrm{d}\mathbb{P}\otimes\mu$$

$$\leq \frac{1}{\varepsilon}\left[\int (|\xi_n^\phi| - h)^+ \, \mathrm{d}\mathbb{P}\otimes\mu + \int (h - \tilde{g})^+ \, \mathrm{d}\mathbb{P}\otimes\mu\right].$$

(3.2) now follows from (3.5); note that given $h \in \mathscr{L}_1^+(\mathbb{P}\otimes\mu)$, $\int (h-\tilde{g})^+ \, \mathrm{d}\mathbb{P}\otimes\mu$ can be made arbitrarily small (choose any $g > 0$ in $\mathscr{L}_1^+(\mu)$, then apply the theorem of monotone convergence to $\int (h - n\tilde{g})^+ \, \mathrm{d}\mathbb{P}\otimes\mu$). □

Remark. Condition (3.5) is satisfied if for all $\phi \in \Phi$ the sequence $\{\xi_n^\phi : n \in \mathbb{N}\}$ is $\mathbb{P}\otimes\mu$-uniformly integrable.

Lemma 3. *Assume that the finite dimensional distributions of ξ_n converge weakly to those of ξ_0 a.e. Then $\{\xi_n : n \in \mathbb{N}\}$ is Φ-weakly tight, if only*

$$\forall \phi \in \Phi: \quad \limsup_{n\to\infty} \int |\xi_n^\phi| \, \mathrm{d}\mathbb{P}\otimes\mu \leq \int |\xi_0^\phi| \, \mathrm{d}\mathbb{P}\otimes\mu < +\infty. \tag{3.6}$$

Proof. Let us verify condition (3.5) of Lemma 2. First note that for $\phi \in \Phi$ (for the notation of $\tilde{h}$, cf. the proof of Lemma 2)

$$\inf_{h \in \mathscr{L}_1^+(\mu)} \int (|\xi_0^\phi| - \tilde{h})^+ \, \mathrm{d}\mathbb{P}\otimes\mu = 0. \tag{3.7}$$

Let $h \in \mathscr{L}_1^+(\mu)$. From Theorem 1 we obtain $\eta_n^h(\cdot, |\phi|) \Rightarrow \eta_0^h(\cdot, |\phi|)$ (for definition of η_n^h cf. (2.5)). Hence (cf. [2, Theorem 5.4])

$$\int \eta_n^h(\omega, |\phi|)\mathbb{P}(\mathrm{d}\omega) \to \int \eta_0^h(\omega, |\phi|)\mathbb{P}(\mathrm{d}\omega), \tag{3.8}$$

since $|\eta_n^h(\omega, |\phi|)| \leq \|h\|_1$. Using (3.6), (3.8), and $|a| = \min\{|a|, b\} + (|a| - b)^+$ for $a \in \mathbb{R}$, $b \geq 0$, we get

$$\begin{aligned}\limsup_{n\to\infty} \int (|\xi_n^\phi| - \tilde{h})^+ \, \mathrm{d}\mathbb{P}\otimes\mu &= \limsup_{n\to\infty} \int |\xi_n^\phi| \, \mathrm{d}\mathbb{P}\otimes\mu - \lim_{n\to\infty} \int \eta_n^h(\cdot, |\phi|) \, \mathrm{d}\mathbb{P} \\ &\leq \int |\xi_0^\phi| \, \mathrm{d}\mathbb{P}\otimes\mu - \int \eta_0^h(\cdot, |\phi|) \, \mathrm{d}\mathbb{P} \\ &= \int (|\xi_0^\phi| - \tilde{h})^+ \, \mathrm{d}\mathbb{P}\otimes\mu.\end{aligned}$$

(3.5) now follows from (3.7). □

C. Let $\psi \in C(S)$ with $\psi > 0$. Let $\Phi = \Phi_\psi$ be the set of all $\mathscr{B}\otimes\mathscr{B}_S$-measurable functions $\phi: T \times S \to \mathbb{R}$ with $\phi(t, \cdot) \in C(S)$ for all $t \in T$ and

$$\|\phi\|_\psi := \sup_{t \in T,\, s \in S} \frac{|\phi(t, s)|}{\psi(s)} < \infty.$$

Lemma 4. $\{\xi_n\colon n\in\mathbb{N}\}$ *is* Φ_ψ*-weakly tight if*

$$\forall\varepsilon>0\colon\quad \inf_{h\in\mathscr{L}_1^+(\mu)}\ \limsup_{n\to\infty}\mathbb{P}\left\{\int(\psi(\xi_n(\omega,\cdot))-h)^+\,\mathrm{d}\mu\geq\varepsilon\right\}=0. \tag{3.9}$$

Remark. Using Lemma 1, (3.4) we can also split condition (3.9) into two parts. In this form we get from Theorem 1 a generalization of the main result of Borovkov and Pecherskii [3].

4. Stochastic processes with paths in $\mathscr{L}_p^E(\mu)$

In part A of this section the results obtained so far are applied to the investigation of weak convergence of stochastic processes with paths in $\mathscr{L}_p^E(\mu)$. It is a well known fact (cf. [4, Theorem 2]) that tightness together with weak convergence of the finite dimensional distributions imply weak convergence of the processes. The theorem stated here is remarkable because the tightness condition can be considerably weakened. In fact, some generalizations of results by Grinblat appear as immediate consequences. The application in part B deals with functionals of the form $f(x)=\int\langle x,g\rangle\,\mathrm{d}\mu$, where $g\in L_q^{E'}$ and the theorem stated can be used to derive a general form of a result by Berman. Finally, in part C, we consider weak convergence of quantile processes taking paths in the Hilbert space $\mathscr{L}_2(0,1)$.

A. Let $S=(E,|\cdot|)$ be a separable Banach space and $\mathscr{B}$ countably generated. Then from [8, III.8.3], $M=\mathscr{L}_p^E(\mu)=\mathscr{L}_p^E(T,\mathscr{B},\mu)$, $1\leq p<\infty$, is separable. $\{\xi_n\colon n\in\mathbb{N}_0\}$ now is a sequence of measurable processes with paths in $\mathscr{L}_p^E(\mu)$. By [4, Theorem 1] (the extension to the E-valued case is possible without greater difficulties—see [5]) the maps $\hat{\xi}_n\colon\Omega\to L_p^E(\mu)$, $\omega\to\hat{\xi}_n(\omega):=\xi_n(\omega,\cdot)$ are $\mathscr{A}-\mathscr{B}(L_p^E(\mu))$-measurable; hence the distributions $\hat{\xi}_n(\mathbb{P})$ of $\hat{\xi}_n$ are well defined probability measures on $(L_p^E(\mu),\mathscr{B}(L_p^E(\mu)))$. Let us say that ξ_n converge weakly to ξ_0 and write $\xi_n\Rightarrow\xi_0$ iff the corresponding image measures $\hat{\xi}_n(\mathbb{P})$ converge weakly, i.e. $\int f\,\mathrm{d}\hat{\xi}_n(\mathbb{P})\to\int f\,\mathrm{d}\hat{\xi}_0(\mathbb{P})$ for all $f\in C_b(L_p^E(\mu))$. Let $\Phi=\Phi_0:=\{\phi_g\colon g\in\mathscr{L}_p^E(\mu)\}$ with

$$\phi_g(t,s):=|s-g(t)|^p,\quad t\in T,\ s\in E.$$

Then $f_{\phi_g}(x)=\int|x(t)-g(t)|^p\mu(\mathrm{d}t)=(\|x-g\|_p^E)^p$, $x\in\mathscr{L}_p^E$, $\phi_g\in\Phi_0$. We now reduce weak convergence of ξ_n to weak convergence of $f_{\phi_g}(\xi_n)$.

Lemma 5. *If the finite dimensional distributions of* ξ_n *converge to those of* ξ_0 *a.e. and if* $\{\xi_n\colon n\in\mathbb{N}\}$ *is* Φ_0*-weakly tight, then* $\xi_n\Rightarrow\xi_0$.

Proof. By Theorem 1 we obtain $(f_{\phi_1}(\xi_n),\ldots,f_{\phi_m}(\xi_n))\Rightarrow(f_{\phi_1}(\xi_0),\ldots,f_{\phi_m}(\xi_0))$ for all $\phi_1,\ldots,\phi_m\in\Phi_0$. Let $\mathscr{F}$ be the vector space generated by the functions $f_\phi^{1/p}$, $\phi\in\Phi_0$; then $f(\xi_n)\Rightarrow f(\xi_0)$ for all $f\in\mathscr{F}$ (cf. [2, Theorem 7.7, Cramer-Wold device]). Since $f_{\phi_g}^{1/p}(x)=\|x-g\|_p^E$, every $f\in\mathscr{F}$ is continuous and $\mathscr{F}$ generates the topology of $L_p^E(\mu)$. Now, by [15, Corollary 1], $f(\xi_n)\Rightarrow f(\xi_0)$ for all $f\in C(L_p^E(\mu))$. But this is equivalent to weak convergence (cf. [2, Theorem 5.2]). $\square$

Theorem 2. *Let $\{\xi_n: n\in\mathbb{N}_0\}$ be a sequence of stochastic processes with paths in $\mathscr{L}_p^E(\mu)$. Then $\xi_n\Rightarrow\xi_0$ provided the finite dimensional distributions of ξ_n converge weakly to those of ξ_0 a.e. and provided one of the following three conditions is satisfied*:

$$\text{The sequence of processes } \{|\xi_n|^p\colon n\in\mathbb{N}\} \text{ is weakly tight;} \tag{4.1}$$

$$\{|\xi_n|^p\colon n\in\mathbb{N}\} \text{ is } \mathbb{P}\otimes\mu\text{-uniformly integrable;} \tag{4.2}$$

$$\limsup_{n\to\infty}\int |\xi_n|^p \,\mathrm{d}\mathbb{P}\otimes\mu \leqslant \int |\xi_0|^p \,\mathrm{d}\mathbb{P}\otimes\mu < +\infty. \tag{4.3}$$

Remarks. 1. By Lemma 1 with $\Phi=\{\phi_0\}\subset\Phi_0$ condition (4.1) is equivalent e.g. to

$$\forall\varepsilon>0:\quad \inf_{h\in\mathscr{L}_1^+(\mu)}\ \limsup_{n\to\infty}\mathbb{P}\left\{\int (|\hat{\xi}_n(\omega)|^p-h)^+\,\mathrm{d}\mu\geqslant\varepsilon\right\}=0. \tag{4.1$'$}$$

2. For finite μ, by [7, II.2.2], condition (4.2) holds true if, for some $\eta>0$,

$$\sup_{n\in\mathbb{N}}\int |\xi_n|^{p+\eta}\,\mathrm{d}\mathbb{P}\otimes\mu<+\infty.$$

In particular we obtain a generalization of Grinblat's results in [11].

3. It is a simple consequence of Fubini's theorem and dominated convergence that condition (4.3) is satisfied if there is some $f\in\mathscr{L}_1^+(\mu)$ with

$$\forall t\in T,\ n\in\mathbb{N}:\ E|\xi_n(t)|^p\leqslant f(t)\quad\text{and}\quad \forall t\in T:\ E|\xi_n(t)|^p\to E|\xi_0(t)|^p$$

(cf. [12] and [13]).

Proof of Theorem 2. By Lemma 5 it suffices to show that each of the conditions (4.1)-(4.3) imply Φ_0-weak tightness of $\{\xi_n: n\in\mathbb{N}\}$. (4.1)$'$$\Rightarrow$(3.2): For (3.2) of Lemma 1 we just show that for $\varepsilon>0$, $\eta>0$, and $g\in\mathscr{L}_p^E(\mu)$ there exists $h\in\mathscr{L}_1^+(\mu)$ with

$$\limsup_{n\to\infty}\mathbb{P}\left\{\int (|\hat{\xi}_n(\omega)-g|^p-2^ph-2^p|g|^p)^+\,\mathrm{d}\mu\geqslant\varepsilon 2^p\right\}\leqslant\eta,$$

since $|g|^p\in\mathscr{L}_1^+(\mu)$. Using $|a+b|^p\leqslant(|a|+|b|)^p\leqslant 2^p(|a|^p+|b|^p)$ for $a,b\in E$, this follows from (4.1)$'$ and

$$\begin{aligned}\int (|\hat{\xi}_n(\omega)-g|^p-2^ph-2^p|g|^p)^+\,\mathrm{d}\mu &\leqslant \int (2^p|\hat{\xi}_n(\omega)|^p-2^ph)^+\,\mathrm{d}\mu\\ &=2^p\int (|\hat{\xi}_n(\omega)|^p-h)^+\,\mathrm{d}\mu.\end{aligned} \tag{4.4}$$

(4.2)$\Rightarrow$(3.5): Using the first part of (4.4) again with μ replaced by $\mathbb{P}\otimes\mu$, we get by (4.2), i.e.

$$\inf_{h\in\mathscr{L}_1^+(\mathbb{P}\otimes\mu)}\ \limsup_{n\to\infty}\int (|\xi_n|^p-h)^+\,\mathrm{d}\mathbb{P}\otimes\mu=0, \tag{4.2$'$}$$

condition (3.5) of Lemma 2. (4.3)$\Rightarrow$(3.5): Applying Lemma 3 with $\Phi=\{\phi_0\}$ $\phi_0(t,s)=|s|^p$ gives (4.2)$'$, hence again (3.5) of Lemma 2. $\square$

B. Let $S=(E,|\cdot|)$ be any separable Banach space with dual E'. We write $\langle s,s'\rangle:=s'(s)$ for $s\in E$, $s'\in E'$. Note that $(s,s')\to\langle s,s'\rangle$ is $\mathscr{B}_E\otimes\mathscr{B}_{E'}$-measurable. Let $M=\mathscr{L}_p^E(\mu)$ with $1<p<\infty$ and let $\Phi=\Phi_1:=\{\phi_g: g\in\mathscr{L}_q^{E'}(\mu)\}$, where $p^{-1}+q^{-1}=1$ and

$$\phi_g(t,s):=\langle s,g(t)\rangle,\quad t\in T,\ s\in E.$$

Then $f_{\phi_g}(x)=\int\langle x(t),g(t)\rangle\mu(\mathrm{d}t)$, $x\in\mathscr{L}_p^E(\mu)$, $\phi_g\in\Phi_1$. For simplicity the following theorem is stated for a single $\phi\in\Phi_1$.

Theorem 3. *Let $\{\xi_n: n\in\mathbb{N}_0\}$ be stochastic processes with paths in $\mathscr{L}_p^E(\mu)$. Then $f_\phi(\xi_n)\Rightarrow f_\phi(\xi_0)$ for all $\phi\in\Phi_1$, i.e.*

$$\int\langle\xi_n(\cdot,t),g(t)\rangle\mu(\mathrm{d}t)\ \Rightarrow\ \int\langle\xi_0(\cdot,t),g(t)\rangle\mu(\mathrm{d}t)$$

for all $g\in\mathscr{L}_q^{E'}(\mu)$, provided the finite dimensional distributions of ξ_n converge to those of ξ_0 a.e. and provided the following condition is satisfied

$$\inf_{N>0}\limsup_{n\to\infty}\mathbb{P}\{\|\xi_n(\omega,\cdot)\|_p^E>N\}=0. \tag{4.5}$$

Proof. We shall verify condition (3.2) of Lemma 1 with $\Phi=\Phi_1$. Let $\phi_g\in\Phi_1$ and $\varepsilon>0$ be given. Use again the notation $\hat{\xi}_n(\omega):=\xi_n(\omega,\cdot)$. Since $\|\hat{\xi}_n(\omega)\|_p^E=\|\,|\hat{\xi}_n(\omega)|\,\|_p$ and $|\langle\xi_n(\omega,t),g(t)\rangle|\leq|\xi_n(\omega,t)|\cdot|g(t)|$ let without loss $E=\mathbb{R}$, $g\geq 0$, and $\xi_n\geq 0$. Now let $a>0$ and $B\in\mathscr{B}$ with $\mu(B)<\infty$. Then for $h\in\mathscr{L}_1^+(\mu)$

$$\int(g\cdot\hat{\xi}_n(\omega)-h)^+\,\mathrm{d}\mu\leq\int_{\{g>a\}\cup(T-B)}g\cdot\hat{\xi}_n(\omega)\,\mathrm{d}\mu$$
$$+\int_{B\cap\{g\leq a\}}(g\cdot\hat{\xi}_n(\omega)-h)^+\,\mathrm{d}\mu.$$

By Hölder's inequality

$$\int_{\{g>a\}\cup(T-B)}g\cdot\hat{\xi}_n(\omega)\,\mathrm{d}\mu\leq(\|g\cdot 1_{\{g>a\}}\|_q+\|g\cdot 1_{T-B}\|_q)\|\hat{\xi}_n(\omega)\|_p.$$

Since $(a-N)^+\leq a^p/N^{p-1}$ for $a\geq 0$, $N>0$,

$$\int_{B\cap\{g<a\}}(g\cdot\hat{\xi}_n(\omega)-h)^+\,\mathrm{d}\mu\leq\int\frac{a^p}{N^{p-1}}\hat{\xi}_n^p(\omega)\,\mathrm{d}\mu=\frac{a^p}{N^{p-1}}\|\hat{\xi}_n(\omega)\|_p^p$$

for $h:=N\cdot 1_B\in\mathscr{L}_1^+(\mu)$. Combining these inequalities and using $\|\hat{\xi}_n(\omega)\|_p\leq 1+\|\hat{\xi}_n(\omega)\|_p^p$ and (4.5), we get

$$\inf_{h\in\mathscr{L}_1^+(\mu)}\limsup_{n\to\infty}\mathbb{P}\left\{\int(g\cdot\hat{\xi}_n(\omega)-h)^+\,\mathrm{d}\mu\geq\varepsilon\right\}$$
$$\leq\inf_{a>0}\ \inf_{B\in\mathscr{B},\mu(B)<\infty}\ \inf_{N>0}\limsup_{n\to\infty}\mathbb{P}\{(\|g\cdot 1_{\{g>a\}}\|_q$$
$$+\|g\cdot 1_{T-B}\|_q+a^p/N^{p-1})\cdot(1+\|\xi_n(\omega,\cdot)\|_p^p)\geq\varepsilon\}=0.\qquad\square$$

Remarks. 1. For $p=1$, Theorem 3 is not applicable even for finite μ and Ω a singleton set. Counterexample: $(T, \mathscr{B}, \mu)=((0,1), \mathscr{B}(0,1), \lambda)$ λ the Lebesgue measure, $\xi_n(t)=n 1_{(0,1/n)}(t)$, $n \in \mathbb{N}$, and $\xi_0 \equiv 0$. Then conditions (2.4) and (4.5) are satisfied, but for $g \equiv 1$ we obtain $f_{\phi_g}(\xi_n)=1 \neq 0=f_{\phi_g}(\xi_0)$. 2. For $p=\infty$ Theorem 3 remains true, if $\mathbb{P}$ in (4.5) is replaced by the inner measure $\mathbb{P}_*$. Note that $\omega \to \|\xi_n(\omega, \cdot)\|_p^E$ may turn out to be nonmeasurable. 3. Theorem 3 can be used to derive a short proof of a generalized result by Berman [1]. Let μ be finite and let λ denote Lebesgue measure in $\mathbb{R}$. Let $\xi_n\colon \Omega \times T \to \mathbb{R}$, $n \in \mathbb{N}_0$, be $\mathscr{A} \otimes \mathscr{B}$-measurable, such that for the processes

$$\eta_n(\omega, u):=\int \mathrm{e}^{\mathrm{i} u \xi_n(\omega, t)} \mu(\mathrm{d} t)$$

we have $\eta_n(\omega, \cdot) \in \mathscr{L}_2(\lambda)$ for all $\omega \in \Omega$.

Corollary. *If the finite dimensional distribution of the processes ξ_n converge to those of ξ_0 a.e. and if*

$$\inf_{N>0} \limsup_{n \to \infty} \mathbb{P}\{\|\eta_n(\omega, \cdot)\|_2 \geq N\}=0,$$

then, for all $g \in \mathscr{L}_2(\lambda)$,

$$\int g(\xi_n(\cdot, t)) \mu(\mathrm{d} t) \Rightarrow \int g(\xi_0(\cdot, t)) \mu(\mathrm{d} t).$$

Proof. By Lemma 1, (3.1)$\Leftrightarrow$(3.4), together with the Remark at the end of Section 3A and Theorem 1, the f.d.d. of $\{\eta_n\}$ converge weakly a.e. Now the proof is complete by the relation $\int g(\xi_n(\cdot, t)) \mu(\mathrm{d} t)=(1/2\pi) \int g(u) \eta_n(\cdot, u) \lambda(du)$, where g denotes the Fourier transform of g (cf. [1, (2, 4)]), and Theorem 3. $\square$

4. Since the functions f_ϕ, $\phi \in \Phi_1$, generate the (weak) topology $\sigma(\mathscr{L}_p^E(\mu), \mathscr{L}_q^{E'}(\mu))$, by the same argument as in the proof of Lemma 5, we obtain from Theorem 3:

Corollary. *Assume that the conditions of Theorem 3 are in force. Then $f(\xi_n) \Rightarrow f(\xi_0)$ for all $f \in C(\mathscr{L}_p^E(\mu), \sigma(\mathscr{L}_p^E(\mu), \mathscr{L}_q^{E'}(\mu)))$.*

C. Our final application deals with quantile processes. Let $X_1, \ldots, X_n$ be independent identically distributed random variables with common distribution function F and density function f. By Q we denote the quantile function of F defined on $(0,1)$, i.e. $Q(u):=\inf\{t\colon F(t) \geq u\}$ for $u \in(0,1)$. Define the empirical quantile function on $(0,1)$ to be $Q_n(\omega, u)=X_{i,n}(\omega)$, whenever $(i-1)/n \leq u<i/n$ for some $1 \leq i \leq n$, where $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics of $X_1, \ldots, X_n$. For any measurable real valued function w with $\int_\varepsilon^{1-\varepsilon} w^2(u)\,\mathrm{d} u<\infty$ for all $\varepsilon>0$ we define the weighted empirical quantile process by

$$r_n^w(\omega, u):=\begin{cases} n^{1/2} w(u)(Q_n(\omega, u)-Q(u)) & \text{for } u \in(1/(n+1), n/(n+1)), \\ 0 & \text{elsewhere.}\end{cases}$$

Let us consider the following two conditions (cf. [14]):

$$F \text{ has a continuous density quantile function } f(Q(u)) \text{ defined on } (0,1), \tag{4.6}$$

$$\int_0^1 u(1-u)w^2(u)h^2(u)\,\mathrm{d}u < +\infty, \tag{4.7}$$

where $h(u) := 1/f(Q(u))$. There is great interest in determining conditions under which the processes r_n^w converge weakly to a continuous process whB, where B denotes a Brownian bridge on $[0, 1]$ (cf. [14] for further references and discussions). We shall consider weak convergence in the Hilbert space $\mathscr{L}_2(0, 1)$ and give a supplement to Mason's Theorem 3 (cf. [14, p. 245 ff.])

Theorem 4. *In addition to assumptions* (4.6) *and* (4.7) *assume that*

$$E\|r_n^w\|_2^2 \to \int_0^1 u(1-u)w^2(u)h^2(u)\,\mathrm{d}u. \tag{4.8}$$

Then $r_n^w \Rightarrow whB$.

Proof. It is easy to show that $\lambda(\{u: f(Q(u)) > 0\}) = 1$. Then weak convergence of the finite dimensional distributions a.e. follows from [6, p. 12 (1.5.11)]. Conditions (4.8) and (4.7) give condition (4.3) of Theorem 2 and the proof is complete. □

Remark. Mason's processes p_n^w and q_n^w can be treated in a similar way, using Anderson's theorem (cf. [14, Prop. 1]).

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